X- and Y- Intercepts Consideration Algorithm for Solving Linear Programming Problems

Chanisara Prayonghom and Aua-aree Boonperm

Abstract— **In this paper, we propose the technique to construct the relaxed problem for solving linear programming problems by the simplex method in 2 dimensions. The algorithm starts by choosing two constraints in less-than constraints which have lowest positive x-intercept and y-intercept to construct the relaxed problem. Then, the relaxed problem is solved by the primal simplex method without using the artificial variable. After the optimal solution of the relaxed problem is found, to guarantee the solution, the relaxed constraints will be restored and the dual simplex method is performed until the optimal solution is obtained. To show the efficiency of the method, the comparison of the results of this method and the original simplex method is proposed.**

Keywords— **artificial variable, linear programming problem, relaxed problem, simplex method.**

I. INTRODUCTION

The simplex method is the popular method for solving a linear programming problem (LP problem) presented by Dantzig [1]. It starts at a vertex in the feasible region, then it moves the solution along the edge consecutively to a better adjacent corner-point in the feasible region until there is no neighboring vertex that has a better solution. In 1972, Klee and Minty [2] showed that the worst case running time of the simplex method is exponential. However, researchers have investigated to improve the simplex method in several ways such as pivoting rule, elimination the redundant constraint, artificial-free, etc.

In 2007, Arsham [3] proposed the simplex algorithm without using artificial variables. Since artificial variables will be added to the greater-than or equal to constraints, Arsham's algorithm starts without using artificial variables by relaxing greater-than or equal to constraints. Then, the relaxed problem is solved by the simplex method. After the optimal solution of the relaxed problem is found, the relaxed constraints are restored, and the dual simplex method is performed. However, if the problem has only greater-than or equal to constraints, the algorithm will use the perturbation simplex method which is other artificial-free technique to solve it.

In 2014, Boonperm and Sinapiromsaran [4-5] proposed the non-acute constraint relaxation technique that improves the simplex method without using the artificial variables, and it can reduce the start-up time to solve the initial relaxation problem. The algorithm starts by relaxing the non-acute constraints which it can guarantee that the relaxed problem is always feasible. So the relaxed problem can be solved without using artificial variables. After the optimal solution of the relaxed problem is found, the relaxed constraints are restored, and the dual simplex method is used to solve it.

Although Arsham's algorithm can solve the linear programming problem without using artificial variables which can reduce the matrix size in each iteration, but it is not guarantee that the optimal solution is found in the relaxed problem while the non-acute relaxation problem has a chance to obtain the optimal solution. However, the redundant constraints which do not correspond the optimal solution are still considered. Therefore, in this paper, we present the improvement of the simplex method without using the artificial variables by relaxing the greater-than or equal to constraints, then some constraints which is considered by the x-intercept and y-intercept from less-than or equal to constraints are selected to construct the relaxed problem. Since the constraints which have minimum x-intercept and y-intercept can guarantee some edges of the feasible region, the others might be the redundant constraints. If we can construct the relaxed problem which has the edge of the feasible region and ignore the other constraints, then we can obtain the optimal solution rapidly. Since the algorithm starts without using artificial variables and the redundant constraints are relaxed, we can reduce the computational time.

This paper is organized as follows. Section 2 explains the preliminaries and the main idea for our algorithm. The algorithm is presented in section 3 followed by illustrative examples in section 4. The last section presents conclusions and future work.

II. THE MAIN IDEA

In general, for a 2-dimensional linear programming problem can be solved by the graphical method or the simplex method. For the graphical method, if the problem has more constraints, then all constraints are drawn for determining the feasible region which is inconvenient for finding the optimal solution (see Figure 1).

Moreover, some constraints might be the redundant constraints which they are not corresponding to the optimal solution. So, we could relax them before the problem is solved.

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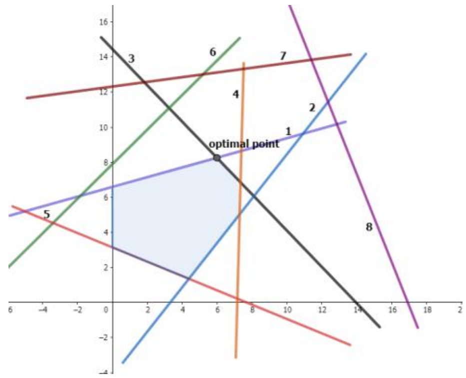

Fig. 1. Feasible region drawn for finding the optimal solution

For the simplex method, it starts at the origin point and moves along the edge to a better adjacent corner-point in the feasible region. However, if the origin point is not a feasible point, then the simplex method cannot start. In this case, artificial variables are introduced to start the simplex algorithm as Figure 2 which it effects to increase the computational time.

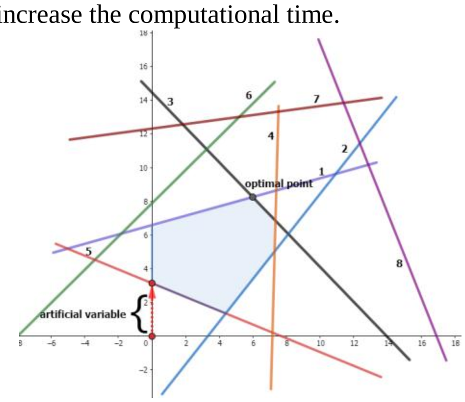

Fig. 2 Artificial variables added for starting the algorithm

Due to the graphical method, we found that we wasted the time to draw the redundant constraints which they do not correspond the optimal solution while the simplex method moves along the edge but it keeps all redundant constraints in each iteration. Therefore, in this paper, we would like to simplify the problem by considering only some edge of the feasible region, and we would like to reduce the time to solve the problem without using artificial variables. At first, we will focus on the 2-dimensional linear programming problem.

Consider a 2-dimensional linear programming problem in the following form:

$$\begin{aligned} \text{Maximize} \quad & z = c_1x_1 + c_2x_2 \\ \text{s.t.} \quad & a_{i1}x_1 + a_{i2}x_2 \le b_i \ , i \in \text{L} \\ & a_{j1}x_1 + a_{j2}x_2 \ge b_j \ , j \in \text{G} \\ & x_1, x_2 \ge 0 \end{aligned} \tag{1}$$

where c_i is the coefficients of the objective function for $i = 1,2$,

a_{ij} is the coefficient of the constraints for $i = 1,2$ and $j = 1,2,...,m$,

x_i is the decision variables for $i = 1,2$,

b_j is the right-hand-side for $j = 1,2,...,m$

and $|\text{G}|+|\text{L}| = m$, $\text{b}_i \ge 0$ for all $i = 1,...,m$.

Since the feasible region of the problem (1) is in the first quadrant and artificial variables are added especially constraints in G. From Figure 1, we will relax constraints in G first and focus on the constraints in L as Figure 3.

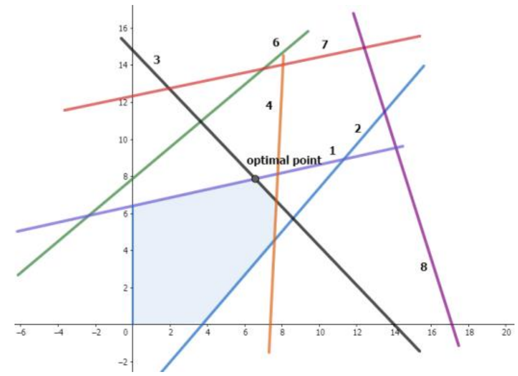

Fig. 3. Relaxing constraints in G

From Figure 3, we found that constraints which have a minimum positive x- and y-intercepts are the edge of the feasible region. Therefore, we will separate the constraints in L into three groups:

$$\text{PP} = \{i \in \mathbf{L} \mid a_{i1} > 0 \text{ and } a_{i2} > 0\},$$

$$\text{NP} = \{i \in \mathbf{L} \mid a_{i1} \le 0 \text{ and } a_{i2} > 0\},$$

and

$$\text{PN} = \{i \in \mathbf{L} \mid a_{i1} > 0 \text{ and } a_{i2} \le 0\}.$$

Constraints in mean that they can bound the feasible region. If , then we can guarantee that the problem has optimal solution. So, we will select some constraints in to construct the relaxed problem. For constraints in , they have only positive y-intercepts. So, we might choose one of them to construct the relaxed problem. Similarly, constraints in have only positive x-intercepts. Therefore, one of them might be chosen.

Therefore, constraints which have minimum x-intercept and y-intercept might form the edge of the feasible region. For guarantee the problem is bounded, one of selected constraints must be constraint in .

For Figure 3, constraints in L are separated as , and .The constraints which have minimum positive x-and y- intercepts are 1 and 2. Since the constraints 1 and 2 are not in , we will pick the constraint 3 which have minimum x-and y-intercepts in for constructing the relaxed problem which drawn as the following figure.

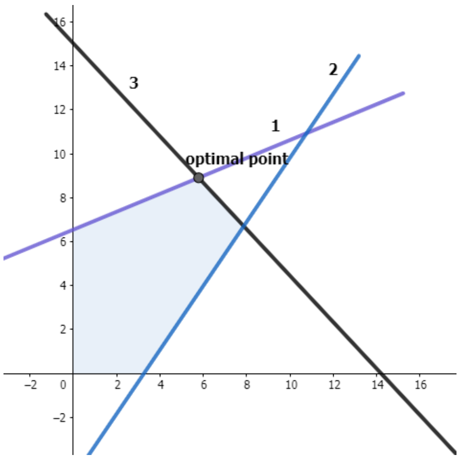

Fig. 4. Feasible region of the relaxed problem

From Figure 4, the relaxed problem has only three constraints, and we can solve by the simplex method without artificial variables.

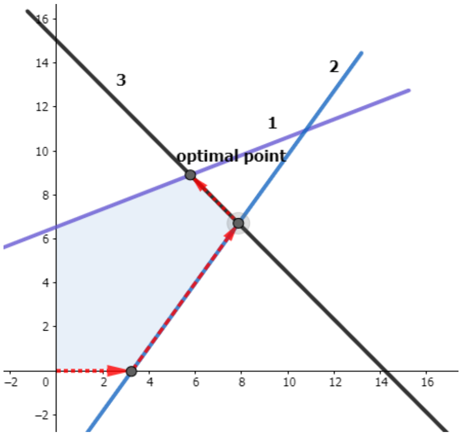

Fig. 5.the simplex method moving in the relaxed problem

From Figure 5, the simplex method can start at the origin point, then it moves along the edge until the optimal point is found. If the current solution is not satisfied some constraints, then we will return the unsatisfied constraints back to the relaxed problem, and the dual simplex can be performed.

Therefore, for any problems which L is not empty, we can construct the relaxed problem which is considered by the x- and y- intercepts to improve the simplex method without using artificial variables. Since some are selected constraints by considering x-and y-intercepts, we will call the algorithm as x-and y- intercepts consideration algorithm or XY-ICA. Next, we present this algorithm in details.

III. THE PROPOSED ALGORITHM

X-and Y-intercepts consideration algorithm (XY-ICA) has steps as follows:

Step 0: Let $\mathbf{G}=\{i \mid \sum_{j=1}^{2} a_{ij}x_{ij} \geq b_i; i=1,\ldots,m\}$,

$\mathbf{L}=\{i \mid \sum_{j=1}^{2} a_{ij}x_{ij} \leq b_i; i=1,\ldots,m\}$,

$\mathrm{PP}=\{i \in \mathrm{L} \mid a_{i1}>0 \text{ and } a_{i2}<0\}$,

$\mathrm{NP}=\{i \in \mathrm{L} \mid a_{i1}\leq 0 \text{ and } a_{i2}>0\}$,

and $\mathrm{PN}=\{i \in \mathrm{L} \mid a_{i1}>0 \text{ and } a_{i2}\leq 0\}$.

Let $M_{ij}=\left\{\frac{b_i}{a_{ij}} \middle| a_{ij}>0 \text{ and } i \in \mathbf{L}, j \in \{1,2\}\right\}$,

and $k_1=\operatorname*{argmin}_{i\in L}\{M_{i1}\}$, $k_2=\operatorname*{argmin}_{i\in L}\{M_{i2}\}$.

Step 1: Construct the relaxed problem which has the constraints k_1 and k_2.

if k_1, $k_2 \notin \mathrm{PP}$, then select one constraint from PP that has minimum positive M_{ij} where $i\in\mathrm{PP}$, $j \in\{1,2\}$ and perform the primal simplex method, go to **step 2**.
else perform the primal simplex method, go to **step2.**

Step 2: If the solution satisfies all constraints, then the optimal solution is found and the algorithm stops.
else restore the constraints which are unsatisfied the current solution and perform the dual simplex method.
else if the optimal solution is found, then the algorithm stops.
else the problem is unbounded.

IV. EXAMPLES

In this section, there are two illustrative examples for showing the efficiency of our algorithm.

Example 1 Consider the following linear programming problem:

$$\begin{aligned}
\text{Maximize } & 2x_1+3x_2 \\
\text{subject to } & x_1+x_2 \geq 2 \\
& -x_1+2x_2 \geq 1 \\
& -2x_1+x_2 \leq 8 \\
& 3x_1-2x_2 \leq 7 \\
& 2x_1-x_2 \leq 9 \\
& 2x_1+x_2 \leq 6 \\
& x_1,x_2 \geq 0
\end{aligned} \tag{2}$$

Step 0: Separate constraints into two groups as follows:

$\mathbf{G}=\{1,2\}$ and $\mathbf{L}=\{3,4,5,6\}$.

Then, constraints in **L** are separated into three groups as below:

$\mathrm{PP}=\{6\}$, $\mathrm{NP}=\{3\}$, and $\mathrm{PN}=\{4,5\}$.

Later, x-and y-intercepts can be computed and summarized as the following table:

TABLE IV: X-AND Y-INTERCEPTS

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	Constraint *i*			
	3	4	5	6
x-intercept	-	2.33	4.5	3
y-intercept	8	-	-	6

From table IV, we get $k_1 = 4$ and $k_2 = 6$. Then, go to step 1.

Since $k_2 = 6$ is in PP, the relaxed problem is written as follows:

$$
\begin{aligned}
\text{Maximize } & 2x_1 + 3x_2 \\
\text{subject } & 3x_1 - 2x_2 \le 7 \\
& 2x_1 + x_2 \le 6 \\
& x_1, x_2 \ge 0
\end{aligned} \tag{3}
$$

After the relaxed problem is converted to the standard form, we get the initial tableau as follows:

TABLE V: **THE INITIAL TABLEAU**

	z	x_1	x_2	x_3	x_4	RHS
z	1	-2	-3	0	0	0
x_3	0	3	-2	1	0	7
x_4	0	2	1	0	1	6

The relaxed problem is solved by the simplex method with 1 iteration, the optimal solution is obtained as the following tableau.

TABLE VI: **THE OPTIMAL TABLEAU**

	z	x_1	x_2	x_3	x_4	RHS
z	1	4	0	0	3	18
x_3	0	7	0	1	2	19
x_2	0	2	1	0	1	6

The optimal solution of the relaxed problem is which satisfies the constraints. Therefore, it is the optimal solution of the original problem.

For this problem, we can compare the computation with the two-phase simplex method as the following table:

TABLE VII: **COMPARISON WITH THE TWO-PHASE METHOD**

The number of iterations		**Size of the matrix**	
XY-ICA method	**two-phase method**	**XY-ICA method**	**two-phase method**
1	4	2×4	6×10

We can see that our algorithm can reduce the computation both of number of iterations and size of matrix for this problem.

Moreover, if we use the graphical method to solved the problem, we can draw the feasible region of the original problem and the relaxed problem as Figure 6 and 7, respectively.

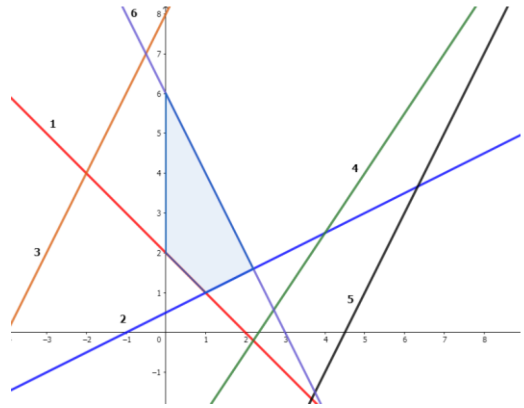

Fig. 6.the original problem

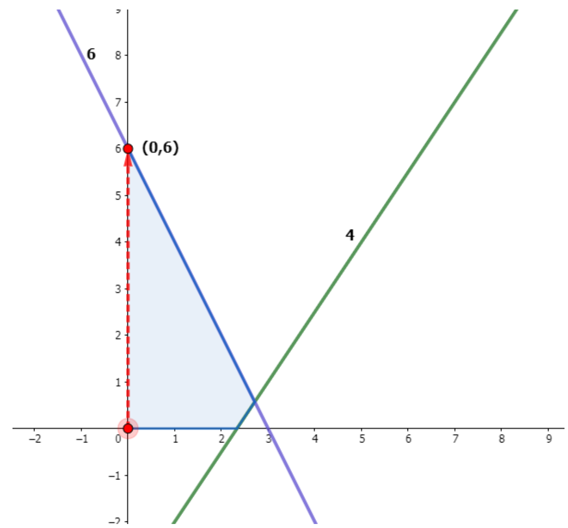

Fig. 7.the relaxed problem

From Figure 7, we can see that the relaxed problem has only two constraints, and the optimal solution can be found in the relaxed problem which satisfies all constraints. Therefore, we can reduce the computation.

Example 2 Consider the following linear programming problem:

$$\begin{aligned} \text{Maximize } & x_1 + 3x_2 \\ \text{subject to } & x_1 - x_2 \le 5 \\ & x_1 + 2x_2 \le 12 \\ & -9x_1 + 4x_2 \le 36 \\ & -2x_1 + 2x_2 \le 6 \\ & 0.5x_1 + 2x_2 \ge 2 \\ & 3x_1 - x_2 \le 3 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1, x_2 \ge 0 \end{aligned} \tag{4}$$

Step 0: Separate constraints into two groups as follows:

$\mathbf{G} = \{5\}$ and $\mathbf{L} = \{1,2,3,4,6,7\}$.

Then, constraints in **L** can be separated into three groups as PP = {2,7}, NP = {3,4}, and PN = {1,6}.

Later, x-and y-intercept can be computed and summarized as the following table:

TABLE VIII: **X-AND Y-INTERCEPTS**

	Constraint *i*					
	1	2	3	4	6	7
x-intercept	5	6	-	-	1	12
y-intercept	-	9	9	3	-	6

From the above table, we get $k_1 = 6$ and $k_2 = 4$. Then, we can go to step 1.

Since k_1, $k_2 \notin$ PP and constraints 2 which is in PP has minimum x-intercept, it is selected to construct the relaxed problem, and it is written as follows:

$$\begin{aligned} \text{Maximize } & x_1 + 3x_2 \\ \text{subject to } & -6x_1 + 2x_2 \le 6 \\ & x_1 - x_2 \le 1 \\ & x_1 + 2x_2 \le 12 \\ & x_1, x_2 \ge 0 \end{aligned} \tag{5}$$

After the relaxed problem is converted to the standard form, we get the initial tableau as follows:

TABLE IX: **THE INITIAL TABLEAU**

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-1	-3	0	0	0	0
x_3	0	-6	2	1	0	0	6
x_4	0	1	-1	0	1	0	1
x_5	0	1	2	0	0	1	12

Then, the relaxed problem is solved by the simplex method with 2 iterations, the optimal solution is obtained as the following tableau.

TABLE X: **THE OPTIMAL TABLEAU**

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	0.071	0	1.428	17.571
x_2	0	0	1	0.071	0	0.428	5.571
x_4	0	0	0	0.214	1	0.285	5.714
x_1	0	1	0	-0.142	0	0.142	0.857

The optimal solution of the relaxed problem is $(x_1, x_2) = (0.857, 5.571)$ which satisfies all constraints. Therefore, it is the optimal solution of the original problem.

For this problem, we can compare the computation with the two-phase simplex method as the following table:

TABLE XI: **COMPARISON WITH THE TWO-PHASE METHOD**

The number of iterations		**Size of the matrix**	
XY-ICA method	**two-phase method**	**XY-ICA method**	**two-phase method**
2	3	3×5	7×10

We can see that our algorithm can reduce the computation both of the number of iterations and size of matrix for this problem.

Moreover, if we use the graphical method to solve the problem, we can draw the feasible region of the original problem and the relaxed problem as Figure 8 and 9, respectively.

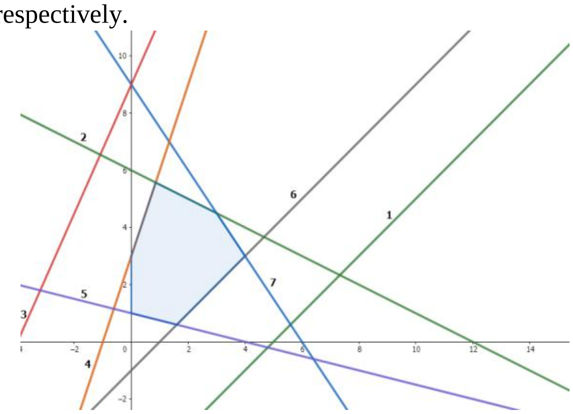

Fig. 8.the original problem

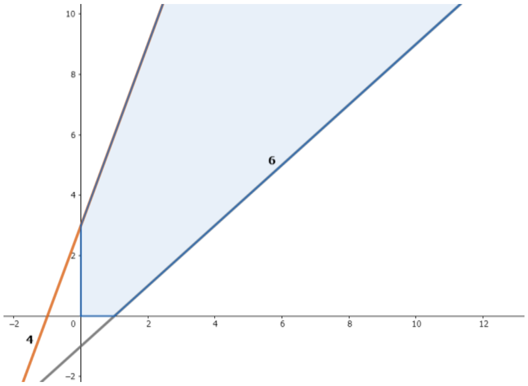

Fig. 9.the relaxed problem without the constraint in PP

From Figure 9, the relaxed problem does not have the constraint in which effects the relaxed problem unbounded. Then, we add the constraint 2 into this problem for guarantee the problem is bounded as Figure 10.

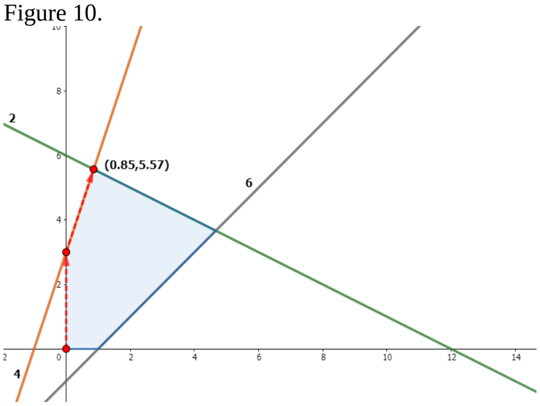

Fig. 10.the relaxed problem with the constraint in PP

From Figure 10, we found that we can solve the original problem by using only three constrains to get the optimal solution.

From example 1 and 2, we can construct the relaxed problem has only few constraints to obtain the optimal solution of the original problem. So, the computation can be reduced.

V. CONCLUSIONS

In this paper, we propose the technique to construct the relaxed problem to simplify the original problem, and the optimal solution can be found in the proposed relaxed problem. The algorithm starts by relaxing the greater-than or equal to constraints, then some constraints which is considered by the x-intercept and y-intercept from less-than or equal to constraints are selected to construct the relaxed problem. Since the relaxed problem consists of some edge of the feasible region, the other might be the redundant constraints which are ignored in the relaxed problem. Therefore, the computational time can be reduced. Due to the proposed examples, we can see that the original problems can be solved by using only two or three constraints to get the optimal solution. However, this work studied in 2-dimensional linear programming problems. We will extend this technique to construct the relaxed problem in n-dimensional linear programming problems..

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