A Simple Technique for Identifying a Solution to a Homogeneous Linear Programming Problem

Tanchanok Phumrachat and Aua-aree Boonperm

Abstract— **In this paper, we propose a simple technique for identifying the solution to a homogeneous linear programming problem by considering a linear combination of all constraints. If the largest coefficient of the linear combination of all constraints is negative or only one zero, then the trivial solution is reported immediately. If the largest and the smallest coefficients are positive, then the solution is also trivial. Otherwise, the largest coefficient of the linear combination of all constraints in the dual problem is considered to find the solution. Since we can identify the solution immediately for some problem by considering only a linear combination, the computations can be reduced extremely.**

***Keywords*— dual problem, homogeneous linear programming problem, linear combination, simplex method, trivial solution .**

I. Introduction

The homogeneous linear programming problem is a problem in the linear programming which right-hand-side vector is a zero vector. The solution of the homogeneous linear programming problem is unbounded or a trivial solution which we can solve by the simplex method created by Dantzig [1]. The simplex method starts when there is a basic feasible variable set. If the basic feasible variable set is not found, artificial variables is added for constructing it. The artificial variable has disadvantages, that are, the computing matrix is bigger, so it is more computational time. The well-known methods that are used to deal with artificial variables are Two-Phase method and Big-M method. For Two-Phase method, it separates into two phases. For Phase I, it finds the minimum of the summation of all artificial variables. After Phase I ends, the basic variable set is found when all artificial variables are zeroes, or infeasibility is reported. For Phase II, it finds the optimal solution by starting at the basic variable set from Phase I. However, it deals with all artificial variables. So, the computational time is increased. Consequently, many researchers have been trying to present methods without using the artificial variable for constructing the basic feasible variable set. In 1997, Arsham [2] constructed the artificial-free simplex algorithm. It consists of two phases. For Phase I, it begins with the empty basic variable set, then the variables are chosen into the basic variable set one by one by considering the coefficient of nonbasic variables in the objective function including the minimum ratio. In Phase I, it can report that any problems are infeasibility or the basic variable set is full. If the basic variable set is full, then Phase II is used to find the optimal solution by the ordinary simplex method. The strong points of this method are the avoidance of artificial variables and protection of cycling for choosing the variables into the basic variable set. However, this algorithm cannot be applied for the homogeneous linear programming problem. It is designed for a positive right-hand-side value. Moreover, in 1998, Enge and Huhn [3] gave a counterexample, in which Arsham's phase I algorithm declares the infeasibility of a feasible problem. Arsham's mistake is that if the variables are chosen into the basic variable set, then other variables with minimum ratio cannot be replaced them. Thus, sometime a basic feasible variable set cannot be found.

In 2015, Gao [4] improved the Arsham's algorithm in two variants. Both start when Arsham's algorithm stops, and it cannot construct the basic feasible variable set, and the nonbasic variable with minimum ratio is not basic variable set. For Variant 1, other nonbasic variables with minimum ratio is allowed to replace the previous variable in the basic variable set. For Variant 2, the linear combination of all unoccupied rows is added, and the nonbasic variable is chosen into the basic variable set by considering the largest coefficient of the added constraint. Although both variants give the basic feasible variable set, this algorithm still has the disadvantage that the variable with no minimum ratio is chosen to enter into the basic variable set from Arsham's algorithm, it must be replaced by the nonbasic variable with minimum ratio, and degeneracy sometime appears. Thus, the number of iterations increases, and it wastes the calculation time for checking that Arsham's algorithm cannot find the solution, then Gao method can start.

Besides the artificial variables are added, for homogeneous linear programming problem, the degeneracy always occurs. So, in this paper, we would like to present the algorithm to solve the homogeneous linear programming problem without using artificial variables and to prevent the degeneracy. By the improvement of Gao and the Two-Phase method, we find that a linear combination of all constraints can be used for constructing the basic feasible variable set, since the coefficients of a linear combination of all constraints is similar to the reduced cost in the initial tableau of Phase I in the Two-Phase method. By the equivalent of the reduced cost in the Two-Phase method and the linear combination of all constraints, we will use the linear combination of all constraints for identifying the solution of the homogeneous linear programming problem. By considering the largest coefficient of

Tanchanok Phumarachat is with Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Thailand.

Aua-aree Boonperm is with Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Thailand.

the linear combination of all constraints, if it is negative, or only one zero, then the solution is trivial. If the largest and the smallest coefficients are positive, then the solution is also trivial. Otherwise, the linear combination of all constraints of the dual problem is considered for constructing the basic variable set. Then, we use the solution of dual problem for answering the original problem. For some problem, this proposed technique can identify the solution of the homogeneous linear programming problem immediately by considering only the linear combination. Thus, the computations can be reduced extremely.

The paper is organized as follows. Section 2 proposes preliminary of the homogeneous linear programming problem. In Section 3, the idea of identifying the solution are presented. In Section 4, the proposed method is established in each step. In Section 5, we show the illustrative examples, and in section 6, we conclude the result of the proposed method..

II. PRELIMINARY

Considers a homogeneous linear programming problem in the following standard form:

$$\begin{aligned} &\text{maximize } c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ &\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ &\qquad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ &\qquad \vdots \qquad \vdots \\ &\qquad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \\ &\qquad x_1, x_2, \ldots, x_n \geq 0 \end{aligned} \tag{1}$$

where n is the number of decision variables,
m is the number of constraints,
c_j is the coefficient of objective function for each $j = 1, \ldots, n$,
a_{ij} are the coefficients of constraints for $i = 1, \ldots, m$, $j = 1, \ldots, n$
and x_j are the nonnegative decision variables for each $j = 1, \ldots, n$.

III. IDEA OF IDENTIFYING A SOLUTION

For the problem (1), we can use the Two-Phase method for finding a solution. The Two-Phase method can start when there is a basic feasible variable set. If it cannot be found easily, then artificial variables are added for constructing the basic variable set. The Two-Phase method is separated into two phases. For Phase I, it finds the basic feasible variable set by minimization of the summation of artificial variables. For Phase II, it uses the simplex method by starting at the basic variable set from Phase I for finding the optimal solution. The Two-Phase method can conclude as the following:

Phase I: Minimize the summation of artificial variables

$$\begin{aligned} &\text{minimize } x_{a_1} + x_{a_2} + \cdots + x_{a_m} \\ &\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{a_1} = 0 \\ &\qquad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{a_2} = 0 \\ &\qquad \vdots \qquad \vdots \\ &\qquad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{a_m} = 0 \\ &\qquad x_1, x_2, \ldots, x_n, x_{a_1}, x_{a_2}, \ldots, x_{a_m} \geq 0 \end{aligned} \tag{2}$$

Fill all coefficients into the simplex tableau:

TABLE I: THE COEFFICIENT TABLEAU OF (2)

	x_1	$\cdots$	x_n	x_{a_1}	$\cdots$	x_{a_m}	RHS
z	0	$\cdots$	0	-1	$\cdots$	-1	0
x_{a_1}	a_{11}	$\cdots$	a_{1n}	1	0	0	0
x_{a_2}	a_{21}	$\cdots$	a_{2n}	0	1	0	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{a_m}	a_{m1}	$\cdots$	a_{mn}	0	0	1	0

Since x_{a_i} are basic variables for $i = 1, \ldots, m$, the reduced cost of x_{a_i} is computed to be zeroes. Thus, the initial tableau is as follows:

TABLE II: THE INITIAL TABLEAU FOR PHASE I

	x_1	$\cdots$	x_n	x_{a_1}	$\cdots$	x_{a_m}	RHS
z	$\sum_{i=1}^{m} a_{i1}$	$\cdots$	$\sum_{i=1}^{m} a_{in}$	0	$\cdots$	0	0
x_{a_1}	a_{11}	$\cdots$	a_{1n}	1	0	0	0
x_{a_2}	a_{21}	$\cdots$	a_{2n}	0	1	0	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{a_m}	a_{m1}	$\cdots$	a_{mn}	0	0	1	0

Since solving the homogeneous linear programming problem by the simplex method will occur degeneracy, for this reason, the problem cannot construct the basic variable set easily. So, we will use the linear combination of all constraints for finding the basic variable set. Since the linear combination of all constraints is similar to the row zero of Phase I in the two-phase method, we use it for constructing the basic variable set.

Let $$\alpha_j = \sum_{i=1}^{m} a_{ij} \quad \text{for each } j = 1, \ldots, n.$$

Then

$$\alpha_1x_1 + \alpha_2x_2 + \ldots + \alpha_nx_n = 0. \tag{3}$$

It is a linear combination of all constraints of (1). By (3), we can use it for identifying a solution of (1). Let

$$\alpha_{\max} = \max\{\alpha_j\} \text{ and } \alpha_{\min} = \min\{\alpha_j\}.$$

By considering the value of $\alpha_{\max}$ and $\alpha_{\min}$, we can distinguish five following cases.

Case 1: $\alpha_{\max} < 0$

That is, all coefficients of (3) are negative, but the value of x_j is only positive or zero for all $j=1,...,n$. Thus, the solution of (1) is x_j be only zero for all $j=1,...,n$, which (3) is satisfied.

Case 2: $\alpha_{\max}=0$ and it is a unique value.
That is the largest coefficient of (3) which is zero, so other coefficients are negative. Therefore, the solution of (1) is $x_j=0$ for all $j=1,...,n$, which (3) is satisfied.

Case 3: $\alpha_{\min}=0$ and it is a unique value.
That is, other coefficients of (3) are positive. Thus, the solution of (1) is $x_j=0$ for all $j=1,...,n$, which (3) is satisfied.

Case 4: $\alpha_{\max}>0$ and $\alpha_{\min}>0$.
That is, all coefficients of (3) are positive. Thus, the solution of (1) is $x_j=0$ for all $j=1,...,n$, which (3) is satisfied.

Case 5: $\alpha_{\max}=0$ or $\alpha_{\min}=0$ and it is not unique.
For this case, we cannot use the linear combination of the original problem for find the solution directly. We will change the original problem to another problem. Associated with each linear programming problem, as (1), there is another linear programming problem called the dual. The dual problem of (1) can be written as the following form:

$$\begin{aligned} &\text{minimize } 0y_1+0y_2+\cdots+0y_m \\ &\text{s.t.}\quad a_{11}y_1+a_{21}y_2+\cdots+a_{m1}y_m \ge c_1 \\ &\qquad a_{12}y_1+a_{22}y_2+\cdots+a_{m2}y_m \ge c_2 \\ &\qquad\qquad \vdots \quad \vdots \\ &\qquad a_{1n}y_1+a_{2n}y_2+\cdots+a_{mn}y_m \ge c_n \\ &\qquad y_1,y_2,...,y_m \text{ unrestricted} \end{aligned} \tag{4}$$

For some constraints, if there is some $c_j<0$ for $j=1,...,n$, then we convert it to positive value by multiplying the constraint by -1. Since $y_1,y_2,...,y_m$ are unrestricted, we can let $y_i=y_i^+-y_i^-$, where $y_i^+,y_i^-\ge 0$ for $i=1,...,m$. Thus, the dual problem in standard form is as follows:

$$\begin{aligned} &\text{minimize } 0(y_1^+-y_1^-)+\cdots+0(y_m^+-y_m^-) \\ &\text{s.t.}\quad a_{11}(y_1^+-y_1^-)+\cdots+a_{m1}(y_m^+-y_m^-)-s_1=c_1 \\ &\qquad a_{12}(y_1^+-y_1^-)+\cdots+a_{m2}(y_m^+-y_m^-)-s_2=c_2 \\ &\qquad\qquad \vdots \quad \vdots \\ &\qquad a_{1n}(y_1^+-y_1^-)+\cdots+a_{mn}(y_m^+-y_m^-)-s_n=c_n \\ &\qquad y_1^+,y_1^-,y_2^+,y_2^-,...,y_m^+,y_m^-,s_1,s_2,...,s_n\ge 0 \end{aligned} \tag{5}$$

By the equivalent of the primal and dual problem, we use the dual problem for finding the solution of the primal problem (1). Since the linear combination consideration cannot give the solution directly in this case, the dual problem of (1), which is (4), is used for solving the problem. Since the solution of the primal homogeneous linear programming problem is only unbounded or trivial, the solution of the dual problem is only infeasible or optimal. If the dual problem is feasible, then the primal problem is optimal with trivial solution. Otherwise, that is, the dual problem is infeasible, the primal problem can be conclude that it is unbounded. We can conclude the solution of primal and dual problem of the primal homogeneous linear programming problem as follows:

TABLE III: THE RELATION OF THE SOLUTION OF PRIMAL AND DUAL PROBLEM

Dual problem		Primal problem
Infeasible	$\Rightarrow$	Unbounded
Feasible	$\Rightarrow$	Trivial

By the above idea, we can use the linear combination of all constraints in the dual problem for identifying the solution of the homogeneous linear programming problem. The steps of the proposed technique are given in detail as follows.Helpful Hints

IV. STEPS OF THE PROPOSED TECHNIQUE

This proposed technique has steps as the following.
Step 1: Convert a homogeneous linear programming problem in the standard form as (1).
Step 2: Compute a linear combination of all constraints as

$$\alpha_1x_1+\alpha_2x_2+...+\alpha_nx_n=0$$

where $\alpha_j=\sum_{i=1}^{m}a_{ij}$ for each $j=1,...,n$.

Step 3: Let $\alpha_{\max}=\max\{\alpha_j\}$ and $\alpha_{\min}=\min\{\alpha_j\}$.

If $\alpha_{\max}<0$, then the solution of (1) is $x_j=0$ for all $j=1,...,n$ and the algorithm stops.
else if $\alpha_{\min}>0$, then the solution of (1) is $x_j=0$ for all $j=1,...,n$ and the algorithm stops.
else if $\alpha_{\max}=0$ or $\alpha_{\min}=0$, and it is unique, then the solution of (1) is $x_j=0$ for all $j=1,...,n$ and the algorithm stops.
else go to Phase I.

Phase I: find the basic variable set.
Step 3.1: Convert the problem (1) to the dual problem as (5).
Step 3.2: Construct an initial tableau, which the basic variable set is empty.
Step 3.3 Add the equation

$$\omega_1^+y_1^++\omega_1^-y_1^-+\omega_2^+y_2^++\cdots+\omega_m^-y_m^-+\delta_1s_1+\cdots+\delta_ns_n=\lambda$$

where $\omega_i^+=\sum_{j=1}^{n}a_{ij}$, $\omega_i^-=-\sum_{j=1}^{n}a_{ij}$ for each $i=1,...,m$, $\delta_i=-1$ for each $i=1,...,n$ and $\lambda=\sum_{j=1}^{n}c_j$ into the row zero of the initial tableau.

Step 3.4: Let $p=\arg\max_{i\in N}\{\omega_i^+,\omega_i^-,\delta_i\}$, where N is a set of index of nonbasic variables, for choosing y_p to enter the basic variable set.

- If $\omega_p < 0$ and $\lambda = 0$, then the solution of the dual problem is $y_i^+ = y_i^- = 0$ for all $i = 1,...,m$ and the solution of the primal problem is trivial. Then, the algorithm stops.

- If $\omega_p \le 0$ and $\lambda > 0$, then the solution of the dual problem is infeasible and the solution of the primal problem is unbounded. Then, the algorithm stops.

- If $\omega_p \ge 0$ and $\lambda \ge 0$, then y_p is chosen to enter into the basic variable set and go to step 3.5.

Step 3.5: Compute the minimum ratio for every $j = 1,...,n$ with $a_{pj} > 0$. Then, bring y_p into a row, which has the minimum ratio, by using Gauss-Jordan row operation and go to step 3.6.

Step 3.6: - If the number of elements of the basic variable set is equal to the number of all constraints in the dual problem, then the optimal solution of the primal problem is trivial.

- Otherwise, go back to step 3.4.

V. ILLUSTRATIVE EXAMPLE

Example 4.1 Consider the following problem:

$$\begin{aligned} \max \quad & x_1 + 5x_2 + 2x_3 + 4x_4 \\ \text{s.t.} \quad & -3x_1 + 2x_2 - 4x_3 - 3x_4 \ge 0, \\ & x_1 - x_2 + 2x_3 - x_4 = 0, \\ & -2x_1 + 5x_2 - 3x_3 - 2x_4 \ge 0, \\ & 3x_1 - 6x_2 + x_3 + x_4 = 0, \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned}$$

We form (6) into the standard form as (1), we get

$$\begin{aligned} \max \quad & x_1 + 5x_2 + 2x_3 + 4x_4 \\ \text{s.t.} \quad & -3x_1 + 2x_2 - 4x_3 - 3x_4 - s_1 = 0, \\ & x_1 - x_2 + 2x_3 - x_4 = 0, \\ & -2x_1 + 5x_2 - 3x_3 - 2x_4 - s_2 = 0, \\ & 3x_1 - 6x_2 + x_3 + x_4 = 0, \\ & x_1, x_2, x_3, x_4, s_1, s_2 \ge 0. \end{aligned} \tag{7}$$

Since the problem (7) is in the standard form, we get the linear combination of all constraints

$$-x_1 + 0x_2 - 4x_3 - 5x_4 - s_1 - s_2 = 0. \tag{8}$$

So $\alpha_1 = -1,\ \alpha_2 = 0,\ \alpha_3 = -4,\ \alpha_4 = -5,\ \alpha_5 = -1,$ and $\alpha_6 = -1$. Then, $\alpha_{\max} = 0$. Since the largest coefficient of (8) is zero and it is unique. Therefore, the solution of (6) is $x_1 = x_2 = x_3 = x_4 = 0$.

Example 4.2 Consider the following problem:

$$\begin{aligned} \max \quad & 2x_1 + x_2 + 4x_3 \\ \text{s.t.} \quad & -x_1 + 2x_2 - x_3 = 0, \\ & 3x_1 - x_2 - 2x_3 = 0, \\ & -2x_1 - x_2 + 3x_3 = 0, \\ & x_1, x_2, x_3 \ge 0. \end{aligned} \tag{9}$$

Since the problem (9) is in the standard form, we get the linear combination of all constraints

$$0x_1 + 0x_2 + 0x_3 = 0 \tag{10}$$

We have $\alpha_1 = 0,\ \alpha_2 = 0,$ and $\alpha_3 = 0$. We see that, the largest coefficients of (10) is zero which it is not unique. Thus, we cannot identify the solution by considering only the linear combination of the primal problem. So, the dual problem of (9) is used for finding the solution of the primal problem.

Phase I: find the basic variable set

The dual problem of the problem (9) is

$$\begin{aligned} \min \quad & 0y_1 + 0y_2 + 0y_3 \\ \text{s.t.} \quad & -y_1 + 3y_2 - 2y_3 \ge 2, \\ & 2y_1 - y_2 - y_3 \ge 1, \\ & -y_1 - 2y_2 + 3y_3 \ge 4, \\ & y_1, y_2, y_3 \text{ unrestricted} \end{aligned} \tag{11}$$

Since y_1, y_2, y_3 are unrestricted, we let $y_i = y_i^+ - y_i^-,\ y_i^+, y_i^- \ge 0$ for $i = 1,2,3$. Thus, the standard form of dual problem of (11) can be written as

$$\begin{aligned} \min \quad & 0(y_1^+ - y_1^-) + 0(y_2^+ - y_2^-) + 0(y_3^+ - y_3^-) \\ \text{s.t.} \quad & -y_1^+ + y_1^- + 3y_2^+ - 3y_2^- - 2y_3^+ + 2y_3^- - s_1 = 2, \\ & 2y_1^+ - 2y_1^- - y_2^+ + y_2^- - y_3^+ + y_3^- - s_2 = 1, \\ & -y_1^+ + y_1^- - 2y_2^+ + 2y_2^- + 3y_3^+ - 3y_3^- - s_3 = 4, \\ & y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^-, s_1, s_2, s_3 \ge 0. \end{aligned} \tag{12}$$

Construct the initial tableau and add the linear combination of all constraints in the row zero as the following:

TABLE IV: THE INITIAL TABLEAU OF (12)

	y_1^+	y_1^-	y_2^+	y_2^-	y_3^+	y_3^-	s_1	s_2	s_3	RHS
ω_j	0	0	0	0	0	0	-1	-1	-1	7
?	-1	1	3	-3	-2	2	-1	0	0	2
?	2	-2	-1	1	-1	1	0	-1	0	1
?	-1	1	-2	2	3	-3	0	0	-1	4

Consider the linear combination of row zero in the initial tableau,

$$0(y_1^+ + y_1^- + y_2^+ + y_2^- + y_3^+ + y_3^-) - s_1 - s_2 - s_3 = 7 \tag{13}$$

We have $\omega_1^+ = \omega_1^- = \omega_2^+ = \omega_2^- = \omega_3^+ = \omega_3^- = 0,\ \delta_1 = \delta_2 = \delta_3 = -1$ and $\lambda = 7$ .Thus, $p = \arg\max_{i \in N}\{\omega_i^+, \omega_i^-, \delta_i\} = 1$.

Since $\omega_1^+ = 0$ and $\lambda = 7 > 0$, the solution of (11) is infeasible. Therefore, the solution of (9) is unbounded.

Example 4.3 Consider the following problem:

$$\begin{aligned} \max \quad & 3x_1 + x_2 + 2x_3 \\ \text{s.t.} \quad & -x_1 - 3x_2 + 2x_3 = 0, \\ & -3x_1 + 2x_2 - x_3 = 0, \\ & 4x_1 + x_2 - x_3 = 0, \\ & x_1, x_2, x_3 \ge 0. \end{aligned} \tag{14}$$

Since the problem (14) is in the standard form, we get the linear combination of all constraints

$$0x_1 + 0x_2 + 0x_3 = 0 \qquad (15)$$

We have $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$. We see that, the largest coefficients of (15) is zero which it is not unique. Thus, we cannot identify the solution by considering only the linear combination of the primal problem. So, the dual problem of (14) is used for finding the solution of the primal problem.

Phase I: find the basic variable set

The dual problem of (14) is

$$\begin{aligned}
\min \quad & 0y_1 + 0y_2 + 0y_3 \\
\text{s.t.} \quad & -y_1 - 3y_2 + 4y_3 \ge 3, \\
& -3y_1 + 2y_2 + y_3 \ge 1, \\
& 2y_1 - y_2 - y_3 \ge 2, \\
& y_1, y_2, y_3 \ \text{unrestricted.}
\end{aligned} \qquad (16)$$

Since y_1, y_2, y_3 are unrestricted, we let $y_i = y_i^+ - y_i^-$, $y_i^+, y_i^- \ge 0$ for $i = 1,2,3$. Thus, the standard form of dual problem of (16) can be written as

$$\begin{aligned}
\min \quad & 0(y_1^+ - y_1^-) + 0(y_2^+ - y_2^-) + 0(y_3^+ - y_3^-) \\
\text{s.t.} \quad & -y_1^+ + y_1^- - 3y_2^+ + 3y_2^- + 4y_3^+ - 4y_3^- - s_1 = 3, \\
& -3y_1^+ + 3y_1^- + 2y_2^+ - 2y_2^- + y_3^+ - y_3^- - s_2 = 1, \\
& 2y_1^+ - 2y_1^- - y_2^+ + y_2^- - y_3^+ + y_3^- - s_3 = 2, \\
& y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^-, s_1, s_2, s_3 \ge 0.
\end{aligned} \qquad (17)$$

Construct the initial tableau and add the linear combination of all constraints in the row zero as the following:

TABLE V: THE INITIAL TABLEAU OF (17)

	y_1^+	y_1^-	y_2^+	y_2^-	y_3^+	y_3^-	s_1	s_2	s_3	RHS
ω_j	-2	2	-2	2	4	-4	-1	-1	-1	6
?	-1	1	-3	3	4	-4	-1	0	0	3
?	-3	3	2	-2	1	-1	0	-1	0	1
?	2	-2	-1	1	-1	1	0	0	-1	2

Consider the linear combination of row zero in the initial tableau,

$$-2y_1^+ + 2y_1^- - 2y_2^+ + 2y_2^- + 4y_3^+ - 4y_3^- - s_1 - s_2 - s_3 = 6. \qquad (18)$$

We have $\omega_1^+ = -2$, $\omega_1^- = 2$, $\omega_2^+ = -2$, $\omega_2^- = 2$, $\omega_3^+ = 4$, $\omega_3^- = -4$, $\delta_1 = \delta_2 = \delta_3 = -1$ and $\lambda = 6$. Thus, $p = \arg\max_{i \in N}\{\omega_i^+, \omega_i^-, \delta_i\} = 5$.

Since $\omega_3^+ = 4 > 0$ and $\lambda = 6 > 0$, the variable y_3^+ is chosen to enter into the basic variable set and compute the minimum ratio for selecting a row. We see that the minimum ratio is in the first row, so y_3^+ is entered into the first row by using Gauss-Jordan row operation. Then, the tableau updates as follows:

TABLE VI: ENTER y_3^+ INTO THE BASIC VARIABLE SET BY ROW OPERATION

	y_1^+	y_1^-	y_2^+	y_2^-	y_3^+	y_3^-	s_1	s_2	s_3	RHS
ω_j	-1	1	1	-1	0	0	0	-1	-1	3
y_3^+	-0.25	0.25	-0.75	0.75	1	-1	-0.25	0	0	0.75
?	-2.75	2.75	2.75	-2.75	0	0	0.25	-1	0	0.25
?	1.75	-1.75	-1.75	1.75	0	0	-0.25	0	-1	2.75

But the number of elements in the basic variable set is not equal to the number of all constraints of (16). Next, we find another nonbasic variable into the basic variable set.

Consider the linear combination of row zero in the current tableau,

$$-y_1^+ + y_1^- + y_2^+ - y_2^- + 0y_3^+ + 0y_3^- + 0s_1 - s_2 - s_3 = 3 \qquad (19)$$

We have $\omega_1^+ = -1$, $\omega_1^- = 1$, $\omega_2^+ = 1$, $\omega_2^- = -1$, $\omega_3^+ = \omega_3^- = \delta_1 = 0$, $\delta_2 = \delta_3 = -1$ and $\lambda = 3$. Thus, $p = \arg\max_{i \in N}\{\omega_i^+, \omega_i^-, \delta_i\} = 2$.

Since $\omega_1^- = 1 > 0$ and $\lambda = 3 > 0$, the variable y_1^- is chosen to enter into the basic variable set. The tableau updates as

TABLE VII: ENTER y_1^- INTO THE BASIC VARIABLE SET BY ROW OPERATION

	y_1^+	y_1^-	y_2^+	y_2^-	y_3^+	y_3^-	s_1	s_2	s_3	RHS
ω_j	0	0	0	0	0	0	-0.09	-0.63	-1	2.91
y_3^+	0	0	-1	1	1	-1	-0.27	0.09	0	0.73
y_1^-	-1	1	1	-1	0	0	0.09	-0.36	0	0.09
?	0	0	0	0	0	0	-0.09	-0.63	-1	2.91

Consider the linear combination of row zero in the current tableau,

$$-0.09s_1 - 0.63s_2 - s_3 = 2.91 \qquad (20)$$

We have $\omega_1^+ = \omega_1^- = \omega_2^+ = \omega_2^- = \omega_3^+ = \omega_3^- = 0$, $\delta_1 = -0.09$, $\delta_2 = -0.63$, $\delta_3 = -1$ and $\lambda = 2.91$. Thus,

$$p = \arg\max_{i \in N}\{\omega_i^+, \omega_i^-, \delta_i\} = 1.$$

Since $\omega_1^+ = 0$ and $\lambda = 2.91 > 0$, the solution of (16) is infeasible. Thus, the solution of (14) is unbounded.

In example 4.1, we can identify the solution rapidly by only the linear combination of all constraints. Thus, we do not waste the computation time in the simplex method and degeneracy does not occur.

In example 4.2, since the linear combination of all constraints of the original problem cannot identify the solution promptly, the dual problem is used for finding the solution. But we only consider the linear combination of all constraints of the dual problem, the algorithm can report immediately that the solution of dual problem is infeasible. By the relationship of the primal and dual problem, we can used it for identifying the solution of the original problem. Therefore, the solution of the original problem is unbounded.

In example 4.3, we cannot identify the solution promptly by considering only the linear combination of all constraints of the original problem. Thus, the dual problem is considered for finding the solution. We construct the basic variable set at the beginning by using phase I, but the basic variable set is not full. Then, the solution of the dual problem is infeasible. Therefore, we conclude that the solution of the original problem is unbounded by the relationship of the primal and dual problem.

VI. Conclusion

The proposed technique can identify a solution to a homogeneous linear programming problem by considering a linear combination of all constraints in a standard form. For the largest coefficient of the linear combination of all constraints is negative or only one zero, the solution is trivial. If the largest and the smallest coefficients are positive, the solution is also trivial. We can answer the solution of the homogeneous problem quickly. Thus, the computations can be reduced extremely. But another case cannot identify the solution promptly, we use the dual problem for finding the solution of the original problem. For this case, we still waste time for constructing the basic variable set at the beginning, then the solution of the original problem can be reported. Thus, it is slowly for identifying the solution. However, the degeneracy cannot occur.

Acknowledgment

This paper was funded by the Faculty of Science and Technology, Thammasat University, Thailand. We would like to thank our faculty for supporting.

References

[1] G. Dantzig, Linear Programming and tensions, Princeton University Press, New Jersey, (1968).

[2] H. Arsham, An artificial-free simplex type algorithm for general LP models, Math. Comput. Model. 25 (1997) 107–123. https://doi.org/10.1016/S0895-7177(97)00043-5 https://doi.org/10.1016/S0895-7177(96)00188-4

[3] A. Enge, P. Huhn, A counterexample to H. Arsham's "initialization of the simplex algorithm: an artificial-free approach", SIAM Rev. Online 40(4) (1998) 1–6.

[4] Pei-wang Gao, Improvement and its computer implementation of an artificial-free simplex-type algorithm by Arsham, Applied Mathematics and Computation 263 (2015) 410–415. https://doi.org/10.1016/j.amc.2015.04.077

Tanchanok Phumrachat was born in Angthong, Thailand, in 1997. She received the B.Sc. degree in mathematics from Thammasat University, Pathumthani, Thailand, in 2015. Now she is studying the M.Sc. degree in mathematics from Thammasat University.

Aua-aree Boonperm was born in Surin, Thailand, in 1981. She received the B.Sc. degree in Mathematics from Khon Kaen University, Thailand, in 2004, M.Sc. degree in Computational Science from Chulalongkorn University, Thailand, in 2007, and Ph.D. degree in Applied Mathematics and Computational science from Chulalongkorn University, Thailand, in 2014.

Now she works at the Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University. Her current research interests include Linear Programming, Operations Research, and Optimization.